Palatini $f(\mathcal{R},\mathcal{L}_m,\mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity and its Born-Infeld semblance

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We investigate Palatini $f(\mathcal{R},\mathcal{L}_m,\mathcal{R}_{\mu\nu}T^{\mu\nu})$ modified theories of gravity. As such, the metric and affine connection are treated as independent dynamical fields, and the gravitational Lagrangian is made a function of the Ricci scalar $\mathcal{R}$, the matter Lagrangian density $\mathcal{L}_m$, and a "matter-curvature scalar" $\mathcal{R}_{\mu\nu}T^{\mu\nu}$. The field equations and the equations of motion for massive test particles are derived, and we find that the independent connection can be expressed as the Levi-Civita connection of an auxiliary, energy momentum–dependent metric that is related to the physical metric by a matrix transformation. Similar to metric $f(\mathcal{R},T,\mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity, the field equations impose the nonconservation of the energy-momentum tensor, leading to the appearance of an extra force on massive test particles. We obtain the explicit form of the field equations for massive test particles in the case of a perfect fluid and an expression for the extra force. The nontrivial modifications to scalar fields and both linear and nonlinear electrodynamics are also considered. Finally, we detail the conditions under which the present theory is equivalent to the Eddington-inspired Born-Infeld theory of gravity.

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I. INTRODUCTION

Observations of the cosmic microwave background (CMB) [1] and direct measurements of the light curves from several hundred type Ia supernovae [2] suggest that the Universe is currently undergoing a phase of late-time, accelerated expansion. While the physics underlying this phenomenon remain unsettled, at least one thing is certain: the acceleration is either a trait of the gravitational interaction itself, or it is a gravitational manifestation of something else (dark energy). By and large, the copious models of the former type derive from revisions to the Einstein-Hilbert action

$$\mathcal{S}_{\mathrm{EH}} = \frac{1}{2\kappa_{\mathrm{E}}}\int \mathrm{d}^4x\sqrt{-g}\mathcal{R}, \tag{1}$$

where κ_{E} is the Einstein constant, $\mathcal{R}$ is the Ricci scalar, and g is the determinant of the spacetime metric $g_{\mu\nu}$.

Among the most straightforward generalizations of $\mathcal{S}_{\mathrm{EH}}$ are the $f(\mathcal{R})$ models. These constitute a class of higher-order gravity theories in which $\mathcal{S}_{\mathrm{EH}}$ is restyled with terms of higher degree in the scalar curvature. Indeed, the mystery of cosmic expansion can be unraveled in this approach [3]. Some models [4] even appear to avoid the fatal instabilities and acute weak-field constraints that bar many other proposals [5]. Incidentally, the lure of $f(\mathcal{R})$ gravity is broader in application than to just cosmic speed-up. For instance, theories with higher-order curvature invariants show promise as effective first-order approximations to quantum gravity and can encourage quantum and gravitational fields to be well behaved in the ultraviolet regions neighboring curvature singularities [6]. Further $f(\mathcal{R})$ phenomenology has been extensively surveyed in the literature [7].

Interesting extensions of the $f(\mathcal{R})$ models are those theories which include in the action an explicit nonminimal coupling between matter and curvature invariants. A notable subset of these models is so-called $f(\mathcal{R},\mathcal{L}_m)$ gravity ($\mathcal{L}_m$ being the matter Lagrangian density) introduced by Bertolami *et al.* in Ref. [8]. Their model was linear in the nonminimal coupling, which prompted the author of Ref. [9] to study the maximal extension of $\mathcal{S}_{\mathrm{EH}}$ in which $\mathcal{R}$ and $\mathcal{L}_m$ are coupled arbitrarily. Cosmological and astrophysical phenomena have been studied in various $f(\mathcal{R},\mathcal{L}_m)$ frameworks [10], in addition to more general studies into the properties of the theory itself [11].

Generally speaking, nonminimal theories such as $f(\mathcal{R},\mathcal{L}_m)$ gravity do not admit chart transitions that locally transform away the influence of a gravitational field on matter [12]. In turn, the covariant divergence of the energy-momentum tensor is generally nonzero, the motion of test particles is generally nongeodesic (due to the presence of an extra force orthogonal to the 4-velocity [8]), and thus the equivalence principle (EP) is generally violated. Hence, these theories are stringently constrained by tests of the EP. It is important to note, however, that a violation of the EP does not in principle disqualify the specific theory [13].

A set of models related to $f(\mathcal{R},\mathcal{L}_m)$ gravity derives from the case in which the functional dependence on $\mathcal{L}_m$

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manifests via a dependence on the trace T of the energy-momentum tensor. These so-called $f(\mathcal{R}, T)$ models have drawn significant attention and were explicitly introduced by Harko *et al.* in Ref. [14]. However, Poplawski [15] was first to consider a model in which the cosmological constant is a function of T, which is considered a relativistically covariant model for interacting dark energy and which is evidently a subset of the $f(\mathcal{R}, T)$ theory. We note that explicit dependences on T may be induced by quantum effects (e.g., conformal anomalies) or exotic imperfect fluids. The reader is referred to the review [16] for additional $f(\mathcal{R}, \mathcal{L}_m)$ and $f(\mathcal{R}, T)$ phenomenology.

Further extensions to the $f(\mathcal{R}, \mathcal{L}_m)$ and $f(\mathcal{R}, T)$ theories were proposed in Refs. [17,18], in which terms of the form $\mathcal{R}_{\mu\nu}T^{\mu\nu}$, where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor, were incorporated into the $f(\mathcal{R}, T)$ Lagrangian. Instances of this coupling are known to arise in Born-Infeld models of gravity [19] when one Taylor expands the Lagrangian. The cosmological implications of these so-called $f(\mathcal{R}, T, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity theories were surveyed in Refs. [17,18,20], and the criterion to circumvent the Dolgov-Kawasaki instability [21] can be found in Ref. [18]. Moreover, energy conditions and thermodynamic laws in $f(\mathcal{R}, T, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity were considered in Ref. [22]. Finally, it is known that metric $f(\mathcal{R}, T, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity acquires ghostlike instabilities due to the additional $\mathcal{R}_{\mu\nu}T^{\mu\nu}$ coupling [23] and that these instabilities can be avoided with a Palatini or metric-affine variation [24].

The appearance of the $\mathcal{R}_{\mu\nu}T^{\mu\nu}$ coupling in Born-Infeld gravity is the chief motivation for our study. The Born-Infeld models themselves, akin to Born-Infeld electromagnetism, modify the determinantal structure of $\mathcal{S}_{\rm EH}$. Among the many Born-Infeld models, a prominent one is the Eddington-inspired Born-Infeld (EiBI) theory proposed in Ref. [25]. Whereas many $f(\mathcal{R})$ models differ from general relativity (GR) even in vacuum, EiBI does not. Yet in ultraviolet regions, such as near cosmological singularities, EiBI gravity is characterized by curing the geometrical divergences plaguing GR [25]. See Ref. [26] for a recent review on Born-Infeld modifications to gravity.

Importantly, in all of these theories, independent of the details of the modification, one must ultimately choose between two ostensibly similar methods for varying the action: either one treats the metric as the sole dynamical entity and fixes *a priori* the connection to be the Levi-Civita connection of $g_{\mu\nu}$ (the metric formalism), or one regards the metric and affine connections as independent dynamical structures (the metric-affine or Palatini formalisms, depending on whether matter couples to the connection or not, respectively). In GR, the distinction is superfluous as they both lead to the same physics. However, in general, nearly all the aforementioned theories forecast different physics depending on whether the metric and affine structures are handled independently or not. In fact, in some theories, such as EiBI gravity [25] and (already mentioned) $f(\mathcal{R}, T, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity [24], the Palatini formalism will remove ghostlike instabilities that otherwise afflict their metric counterparts. Whether the affine connection is determined by the metric degrees of freedom (d.o.f.) or not is a truly fundamental question and demands experimental investigation.

Though matter couplings to the connection may arise due to quantum gravitational corrections, we shall ignore that possibility here, and so we exercise the Palatini formalism. Studies of Palatini $f(\mathcal{R})$ and $f(\mathcal{R}, T)$ models can be found in Refs. [27] and [28,29], respectively, and more general actions varied *à* la Palatini and metric-affine, including the role of torsion, can be found in Ref. [24]. To the best of our knowledge, no studies of pure Palatini $f(\mathcal{R}, T, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ or Palatini $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity have yet been completed, though indirect pursuits exist (see, e.g., Ref. [24]). In this paper, we shall investigate Palatini $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity, from which Palatini $f(\mathcal{R}, T, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity follows after a simple modification to the field equations. In addition to studying $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity on its own, we ultimately seek the conditions under which our theory corresponds to EiBI gravity.

The present paper is structured as follows. In Sec. II, we vary the $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ action *à* la Palatini and derive the theory's equations of motion and an explicit form for the independent connection. In Sec. III, we survey the bimetric structure of $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity in addition to the nonminimal structure of the field equations. In Sec. IV, we explore various properties of the $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ field equations, including their nonconservation equation, the nongeodesic motion of test particles, the nature of the extra force, the weak-field limit, and the modified Poisson equation. In Sec. V, we derive the $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ field equations for the cases of linear and nonlinear electromagnetic fields as well as canonical scalar fields. Finally, in Sec. VI, we derive the conditions under which the $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ model responds identically to the EiBI theory for specific matter sectors.

In this paper, we shall operate in a four-dimensional spacetime $(\mathcal{M}, g_{\mu\nu}, \Gamma^\alpha_{\mu\nu})$ in which the metric $g_{\mu\nu}$ and connection $\Gamma^\alpha_{\mu\nu}$ will be treated as independent dynamical fields. We shall utilize the metric signature $(-,+,+,+)$ and, where appropriate, adopt the natural system of units in which $c = 8\pi G = 1$.

II. FIELD EQUATIONS OF $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ GRAVITY

The Ricci tensor can be defined solely in terms of the affine connection, and this underpins the Palatini and metric-affine formalisms. Explicitly, the Ricci tensor follows from the Riemann curvature tensor

$$\mathcal{R}^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta} \tag{2}$$

via the contraction $\mathcal{R}_{\mu\nu}(\Gamma)\equiv\mathcal{R}^{\alpha}{}_{\mu\alpha\nu}(\Gamma)$. Only now, one needs to invoke the metric to define the Ricci scalar $\mathcal{R}(g,\Gamma)\equiv g^{\mu\nu}\mathcal{R}_{\mu\nu}(\Gamma)$ and a *matter-curvature scalar* $V(g,\Gamma,\Psi)\equiv\mathcal{R}_{\mu\nu}(\Gamma)T^{\mu\nu}(g,\Psi)$, where $T_{\mu\nu}$ is the symmetric (Hilbert) energy-momentum tensor. As we shall see below in Eq. (5), the energy-momentum tensor is constructed *à* la Palatini so that $T_{\mu\nu}$ depends only on the metric and a set of matter fields Ψ. We note that the symmetry of $g_{\mu\nu}$ and $T_{\mu\nu}$ imposes that only the symmetric part of the Ricci tensor enters into this theory's action. This considerably simplifies the role of torsion in the theory and renders a separate consideration for fermionic matter immaterial [24].

With all this in mind, the action considered in this work bears the form

$$\mathcal{S}[g,\Gamma,\Psi]=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}f(\mathcal{R},\mathcal{L}_m,V)+\mathcal{S}_m[g,\Psi], \quad (3)$$

where κ is a coupling constant with suitable dimensions. Here, the matter Lagrangian density $\mathcal{L}_m$, encoded in both the function $f(\mathcal{R},\mathcal{L}_m,V)=f(\mathcal{R},\mathcal{L}_m,\mathcal{R}_{\mu\nu}T^{\mu\nu})$ and the matter action

$$\mathcal{S}_m[g,\Psi]=\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_m[g,\Psi], \quad (4)$$

is assumed to capture all matter fields Ψ present in $\mathcal{M}$. Moreover, $\mathcal{L}_m$ determines the manifestly symmetric energy-momentum tensor

$$T_{\mu\nu}\equiv-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}, \quad (5)$$

which again is independent of the affine connection in the Palatini formulation.

If we denote by $\delta\mathcal{S}_g$ and $\delta\mathcal{S}_\Gamma$ the variation of Eq. (3) with respect to the metric and connection, respectively, then $\delta\mathcal{S}=\delta\mathcal{S}_g+\delta\mathcal{S}_\Gamma$ with

$$\delta\mathcal{S}_g=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\Big[-\frac{1}{2}fg_{\mu\nu}+f_{\mathcal{R}}\mathcal{R}_{\mu\nu}+f_{\mathcal{L}}\Xi_{\mu\nu} + f_V\Pi_{\mu\nu}-\kappa T_{\mu\nu}\Big]\delta g^{\mu\nu} \quad (6)$$

and

$$\delta\mathcal{S}_\Gamma=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left[(f_{\mathcal{R}}g^{\mu\nu}+f_VT^{\mu\nu})\frac{\delta\mathcal{R}_{\mu\nu}}{\delta\Gamma^{\lambda}_{\alpha\beta}}\right]\delta\Gamma^{\lambda}_{\alpha\beta}. \quad (7)$$

Here, we have introduced the definitions $f_{\mathcal{R}}\equiv\partial_{\mathcal{R}}f$, $f_V\equiv\partial_Vf$, $f_{\mathcal{L}}\equiv\partial_{\mathcal{L}_m}f$ as well as the manifestly symmetric *matter* and *matter-curvature tensors*

$$\Xi_{\mu\nu}\equiv\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}, \quad (8)$$

$$\Pi_{\mu\nu}\equiv\mathcal{R}^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}, \quad (9)$$

respectively. Since $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ are independent fields, $\delta\mathcal{S}=0$ if and only if $\delta\mathcal{S}_g$ and $\delta\mathcal{S}_\Gamma$ vanish separately. In the case of the metric variation (6), $\delta\mathcal{S}_g=0$ implies

$$f_{\mathcal{R}}\mathcal{R}_{\mu\nu}-\frac{1}{2}fg_{\mu\nu}=\kappa T_{\mu\nu}-f_{\mathcal{L}}\Xi_{\mu\nu}-f_V\Pi_{\mu\nu}. \quad (10)$$

This is the $f(\mathcal{R},\mathcal{L}_m,V)$ generalization of Einstein's equation. Its properties shall be explored in the coming sections. We note here, however, that as a consequence of the nonminimal coupling there appear in Eq. (10) strict couplings between matter fields and curvature terms. This is very much unlike GR and other minimally coupled theories in which matter fields are wholly separable from curvature terms such that the field equations may be written in a "curvature = matter" type representation. Ultimately, however, writing the field equations in this way is more for physical tidiness and less for mathematical substance. Hence, the mathematical representation of these equations may as well be chosen such that it facilitates later computation. To this end, we define a curvature-dependent *effective* energy-momentum tensor by

$$\Sigma_{\mu\nu}\equiv T_{\mu\nu}-\frac{f_{\mathcal{L}}}{\kappa}\Xi_{\mu\nu}-\frac{f_V}{\kappa}\Pi_{\mu\nu}, \quad (11)$$

which refashions the field equations (10) into a form similar to those in Palatini $f(\mathcal{R})$ gravity:

$$f_{\mathcal{R}}\mathcal{R}_{\mu\nu}-\frac{1}{2}fg_{\mu\nu}=\kappa\Sigma_{\mu\nu}. \quad (12)$$

The variation with respect to the connection takes more care. We refer the reader to Ref. [24], in which a nearly complete derivation is given. One shall find that $\delta\mathcal{S}_\Gamma=0$ only if

$$\nabla^{(p)}_\sigma[\sqrt{-g}(f_{\mathcal{R}}g^{\mu\nu}+f_VT^{\mu\nu})]=0, \quad (13)$$

where $\nabla^{(p)}$ is the derivative operator associated with the independent connection and which is manifestly distinct from $\nabla^{(g)}$, the covariant derivative compatible with the spacetime metric $g_{\mu\nu}$. The resemblance of Eq. (13) to the companion EiBI field equation will be studied in Sec. VI.

We note that Eq. (13) holds well even in the presence of torsion. This follows from this theory's insensitivity to the projective d.o.f. in projective transformations of the independent connection, which ultimately derives from only the symmetric part of the Ricci tensor entering into the action (3). See Ref. [24] for details.

Together, Eqs. (10) and (13) comprise the field equations for $f(\mathcal{R},\mathcal{L}_m,V)$ gravity. We see in Eq. (13) a natural auxiliary metric ingrained into this theory's mathematical structure, namely, a metric $p_{\mu\nu}$ of which the *inverse*, denoted $p^{\mu\nu}$ [30], satisfies

$$\sqrt{-p}\,p^{\mu\nu} = \sqrt{-g}(f_{\mathcal{R}}g^{\mu\nu} + f_V T^{\mu\nu}), \tag{14}$$

where $p \equiv \det(p_{\mu\nu})$. Evidently, the symmetry of $g^{\mu\nu}$ and $T^{\mu\nu}$ forces $p^{\mu\nu}$ (and hence $p_{\mu\nu}$) to be symmetric. Moreover, $p^{\mu\nu}$ satisfies $\nabla^{(p)}_\sigma(\sqrt{-p}\,p^{\mu\nu}) = 0$ by construction, so $p_{\mu\nu}$ is compatible with $\nabla^{(p)}$, provided the coefficients of the independent connection are the Christoffel symbols in $p_{\mu\nu}$,

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}p^{\alpha\sigma}(\partial_\mu p_{\sigma\nu} + \partial_\nu p_{\mu\sigma} - \partial_\sigma p_{\mu\nu}). \tag{15}$$

Consequently, the independent connection is the Levi-Civita connection in the auxiliary metric $p_{\mu\nu}$. Note also that the determinant p can be computed explicitly with Eq. (14) and the relation $p = \det^{-1}(p^{\mu\nu})$. One finds

$$p = g^2 \det(f_{\mathcal{R}}g^{\mu\nu} + f_V T^{\mu\nu}). \tag{16}$$

We shall apply these formulas to various physical phenomena in the coming sections. But first, we briefly comment on some general characteristics of the field equations.

III. REMARKS ON THE $f(\mathcal{R},\mathcal{L}_m,V)$ FIELD EQUATIONS

As noted above, for theories in which couplings are minimal, the matter fields can in general be placed on one side of the theory's field equation, and the symmetric part of the Ricci tensor will be given solely in terms of $g_{\mu\nu}$. But for nonminimal theories, the matter fields are generally inseparable from the geometry terms, and the symmetric part of the Ricci tensor need not be given solely in terms of $g_{\mu\nu}$. Such is the case for $f(\mathcal{R},\mathcal{L}_m,V)$ gravity, as made evident by the field equations (10) and (13). Other aspects of the present theory's nonminimal character are addressed in this section.

A. Matter and matter-curvature tensors

The matter-curvature tensor $\Pi_{\mu\nu}$ is a hallmark of the present theory's nonminimal coupling. For the sake of computation, it is of interest to write this tensor in a form entirely in terms of the matter Lagrangian and Ricci tensor. To this end, assuming the matter Lagrangian is independent of derivatives of the metric, one can show that Eq. (5) is equivalent to

$$T^{\mu\nu} = \mathcal{L}_m g^{\mu\nu} + 2\frac{\partial\mathcal{L}_m}{\partial g_{\mu\nu}}. \tag{17}$$

Incidentally, this equation has the matter tensor $\Xi_{\mu\nu}$ implicitly built into it,

$$\Xi_{\mu\nu} = \frac{1}{2}(\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}), \tag{18}$$

which we shall find useful later on. Moreover, Eq. (17) facilitates calculating the functional derivative

$$\frac{\delta T^{\alpha\beta}}{\delta g^{\mu\nu}} = g^{\alpha\beta}\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}} + 2\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g_{\alpha\beta}} + \mathcal{L}_m\delta^{(\alpha\beta)}{}_{\mu\nu}, \tag{19}$$

where $\delta^{(\alpha\beta)}{}_{\mu\nu} = \frac{1}{2}(\delta^\alpha{}_\mu\delta^\beta{}_\nu + \delta^\beta{}_\mu\delta^\alpha{}_\nu)$ is the upper symmetric part of the generalized Kronecker symbol (we herein denote symmetrization by parentheses). Combining this result with the definition of the matter-curvature tensor in Eq. (9) implies

$$\Pi_{\mu\nu} = \mathcal{R}\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}} + 2\mathcal{R}_{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g_{\alpha\beta}} + \mathcal{R}_{\mu\nu}\mathcal{L}_m. \tag{20}$$

Another useful identity is

$$\Pi_{\mu\nu} = 2\mathcal{R}_{\lambda(\mu}T^\lambda{}_{\nu)} + \mathcal{R}^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}, \tag{21}$$

which follows from substituting $T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta}$ into the definition (9) of the matter-curvature tensor.

A notable matter sector is that of a perfect fluid (PF), for which we shall take $\mathcal{L}_m = P$ [31], where P is the isotropic pressure of the fluid. The corresponding energy-momentum tensor is

$$T^{(\mathrm{PF})}_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}, \tag{22}$$

where ρ is the energy density of the fluid and the fluid's 4-velocity u^μ satisfies the condition $u_\mu u^\mu = -1$. One can show the pressure P satisfies

$$\delta P = -\frac{1}{2}(\rho + P)u_\mu u_\nu \delta g^{\mu\nu} \tag{23}$$

by using Eq. (17) with $\mathcal{L}_m = P$ and comparing to Eq. (22). Moreover, one has [33]

$$\delta\rho = \frac{1}{2}\rho(g_{\mu\nu} - u_\mu u_\nu)\delta g^{\mu\nu}, \tag{24}$$

which ultimately follows from the conservation of the matter fluid current, $\nabla^{(g)}_\mu(\rho u^\mu) = 0$. Using these formulas appropriately, one shall find

$$\Xi^{(\mathrm{PF})}_{\mu\nu} = -\frac{1}{2}(\rho + P)u_\mu u_\nu \tag{25}$$

and, from Eq. (21) and the identity $\frac{\delta u_\alpha}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\alpha(\mu}u_{\nu)}$,

$$\Pi^{(\mathrm{PF})}_{\mu\nu} = \mathcal{R}^{\lambda}{}_{(\mu}T^{(\mathrm{PF})}_{\nu)\lambda} + \frac{1}{2}\rho\mathcal{R}^{\alpha\beta}u_\alpha u_\beta g_{\mu\nu} - \frac{1}{2}[(2\rho+P)\mathcal{R}^{\alpha\beta}u_\alpha u_\beta + (\rho+P)\mathcal{R}]u_\mu u_\nu. \tag{26}$$

The effective energy-momentum tensor for a perfect fluid then follows from its definition (11). The rather exotic coupling of matter and the 4-velocity to the Ricci tensor in Eq. (26) suggests that the matter-curvature tensor will play a significant role in the field equations in regions of high density, such as within a black hole or in the very early Universe. This is quantitatively similar to EiBI gravity, which has in its field equations a similar $\mathcal{R}_{\mu\nu}T^{\mu\nu}$ coupling that also gives rise to couplings between the Ricci tensor and the 4-velocity of perfect fluids (see Ref. [26] or Sec. VI of this paper). It is natural to hypothesize, then, that $f(\mathcal{R},\mathcal{L}_m,V)$ gravity may be fashionable such that it corresponds to GR in the weak-field regime but then cures the curvature singularities of GR in high density regions—behavior that mimics the preeminent characteristics of EiBI gravity.

B. Auxiliary metric

The introduction of the "natural" auxiliary metric $p_{\mu\nu}$ into the present theory affords a specific bimetric structure to the $f(\mathcal{R},\mathcal{L}_m,V)$ model. In addition to the physical spacetime metric $g_{\mu\nu}$, through which the gravitational observables manifest, there is the auxiliary metric upon which the mathematical edifice of the theory is best supported. This structure is analogous to EiBI gravity wherein there also exists a natural bimetric arrangement [25,26]. In the present theory, however, unlike the minimal nature of EiBI theory, the gravitational Lagrangian has built into it a direct coupling between the matter fields and the auxiliary metric via the scalar curvature $\mathcal{R}$ and the matter-curvature scalar $V \equiv \mathcal{R}_{\mu\nu}T^{\mu\nu}$. This coupling appears through the explicit dependence of the independent connection (15) on $p_{\mu\nu}$ and its inverse. Such a coupling suggests that there is some physical nature tied to the auxiliary metric. But because all physical observables manifest via the spacetime metric, the nonminimal coupling suggest a general link between the auxiliary and spacetime metrics. Obviously, the particulars of this link cannot be properly realized until the details of the nonminimal coupling are known (which necessitates specifying a particular function f). However, a general relationship can be drawn.

A natural link to proffer is that of a conformal relationship, in which $p_{\mu\nu} = \Theta^2 g_{\mu\nu}$ for some real-valued, smooth function Θ defined on $\mathcal{M}$. This approach, however, is consistent only for specific matter sectors [34]. Thus, conformality between $p_{\mu\nu}$ and $g_{\mu\nu}$ fails to capture the general framework we seek. A more general approach, again analogous to EiBI gravity, is to introduce a differentiable *deformation matrix* $\Omega^{\mu}{}_{\nu}$ satisfying

$$p_{\mu\nu} = g_{\mu\lambda}\Omega^{\lambda}{}_{\nu}. \tag{27}$$

In matrix notation, this reads $\mathbf{p} = \mathbf{g}\mathbf{\Omega}$ so that $\mathbf{p}^{-1} = \mathbf{\Omega}^{-1}\mathbf{g}^{-1}$. Direct comparison to Eq. (14) reveals that

$$\mathbf{\Omega}^{-1} = \frac{1}{\sqrt{\Omega}}(f_{\mathcal{R}}\mathbf{I} + f_V \mathbf{g}^{-1}\mathbf{T}), \tag{28}$$

where $\mathbf{I}$ is the identity matrix and $\Omega \equiv \det(\mathbf{\Omega})$ follows from Eq. (16). It is now an algebraic problem to solve for $\mathbf{\Omega}$ and hence $p_{\mu\nu}$, explicitly, following the specification of the matter Lagrangian and the $f(\mathcal{R},\mathcal{L}_m,V)$ model of interest. One subsequently obtains the form of the connection and related curvature terms for the specific theory, and all that remains to resolve a given problem is the differential equations (10). An example of this procedure, in the context of EiBI gravity, can be found in Ref. [26].

C. Likeness to other f theories

The Palatini $f(\mathcal{R},\mathcal{L}_m,V)$ formalism contains as special cases the Palatini $f(\mathcal{R})$ and $f(\mathcal{R},\mathcal{L}_m)$ theories but not in general the Palatini $f(\mathcal{R},T)$ and $f(\mathcal{R},T,V)$ theories. Evidently, Palatini $f(\mathcal{R},\mathcal{L}_m,V)$ and $f(\mathcal{R},T,V)$ gravity correspond only when $\mathcal{L}_m = T$, which is a hefty constraint by which most matter fields do not abide [35]. That said, for matter fields with a vanishing energy-momentum trace (such as electromagnetic fields), the $f(\mathcal{R},\mathcal{L}_m,V)$ model clearly contains the $f(\mathcal{R},T,V)$ model. We say that Palatini $f(\mathcal{R},\mathcal{L}_m,V)$ and $f(\mathcal{R},T,V)$ are *circumstantially equivalent* theories of gravity since their equivalence is such that it holds only for specific matter fields (this notion is made more precise in Sec. VI). There is, however, a simple procedure to obtain Palatini $f(\mathcal{R},T,V)$ gravity from the $f(\mathcal{R},\mathcal{L}_m,V)$ theory for arbitrary matter sectors: merely replace the $f_{\mathcal{L}}\Xi_{\mu\nu}$ term in the field equations (10) by $f_T \frac{\partial T}{\partial g^{\mu\nu}}$, and continue on that way. Since in the Palatini formalism the trace $T \equiv T^{\mu}{}_{\mu}$ is independent of the independent connection, its incorporation into the function f will not affect Eq. (13). In this respect, most results derived herein afford similar mathematical structure to Palatini $f(\mathcal{R},T,V)$ gravity, up to the replacement of all $f_{\mathcal{L}}\Xi_{\mu\nu}$ terms with $f_T \frac{\partial T}{\partial g^{\mu\nu}}$ terms and the subsequent manipulations of those terms. Evidently, the exception to this rule is those results which utilize, in a nontrivial manner, the full entourage of dependencies in the $f(\mathcal{R},\mathcal{L}_m,V)$ model, such as the present theory's circumstantial equivalence to EiBI gravity (see Sec. VI).

IV. PROPERTIES OF THE $f(\mathcal{R},\mathcal{L}_m,V)$ FIELD EQUATIONS

Here, we shall consider various properties of the $f(\mathcal{R},\mathcal{L}_m,V)$ field equations, including their conservation equation, their effect on the motion of massive test particles, and their weak-field limit.

A. Conservation equation

In $f(\mathcal{R}, \mathcal{L}_m, V)$ gravity, matter is nonminimally coupled to curvature. Hence, the covariant divergence of the energy-momentum tensor is not necessarily zero. In this section, we derive an explicit expression for such nonconservation of the energy-momentum tensor. In what follows, we use tildes to decorate tensors which have been transvected by the auxiliary metric $p_{\mu\nu}$.

We begin with the field equations (12) in the form

$$\tilde{G}^{\mu}{}_{\nu} = \frac{1}{f_{\mathcal{R}}}\left[\kappa\tilde{\Sigma}^{\mu}{}_{\nu} + \frac{1}{2}f(\Omega^{-1})^{\mu}{}_{\nu}\right] - \frac{1}{2}\delta^{\mu}{}_{\nu}\tilde{\mathcal{R}}, \tag{29}$$

where $\tilde{G}^{\mu}{}_{\nu}$ is the Einstein tensor raised by $p^{\mu\nu}$ and $(\Omega^{-1})^{\mu}{}_{\nu}$ is $\mathbf{\Omega}^{-1}$ in index notation. Using the definition (14), one finds

$$\tilde{\Sigma}^{\mu}{}_{\nu} = \sqrt{\frac{g}{p}}(f_{\mathcal{R}}\Sigma^{\mu}{}_{\nu} + f_V T^{\mu\lambda}\Sigma_{\lambda\nu}), \tag{30a}$$

$$\tilde{\mathcal{R}} = \sqrt{\frac{g}{p}}(\mathcal{R}f_{\mathcal{R}} + Vf_V). \tag{30b}$$

The condition we seek follows from the contracted Bianchi identities, $\nabla^{(p)}_{\mu}\tilde{G}^{\mu}{}_{\nu} = 0$. All that remains is a straightforward problem in algebra: expand $\Sigma^{\mu}{}_{\nu}$ in Eq. (30a) using its definition (11), then isolate the covariant divergence of $T^{\mu}{}_{\nu}$. We find

$$\kappa\nabla^{(p)}_{\mu}T^{\mu}{}_{\nu} = \sqrt{\frac{p}{g}}\left\{\nabla^{(p)}_{\mu}\left[\sqrt{\frac{g}{p}}\left(f_{\mathcal{L}}\Xi^{\mu}{}_{\nu} + f_V\Pi^{\mu}{}_{\nu} - \frac{f_V}{f_{\mathcal{R}}}T^{\mu\lambda}\Sigma_{\lambda\nu} - \frac{ff_V}{2f_{\mathcal{R}}}T^{\mu}{}_{\nu}\right)\right] - \frac{1}{2}\partial_{\nu}\left(\sqrt{\frac{g}{p}}(f - \mathcal{R}f_{\mathcal{R}} - Vf_V)\right) - \kappa T^{\mu}{}_{\nu}\partial_{\mu}\left(\sqrt{\frac{g}{p}}\right)\right\}. \tag{31}$$

Alternatively, this nonconservation can be expressed in terms of the connection $\nabla^{(g)}$ compatible with the spacetime metric $g_{\mu\nu}$. The relationship between the covariant derivatives $\nabla^{(p)}$ (that defined with the independent connection of the auxiliary metric) and $\nabla^{(g)}$ is the following:

$$\nabla^{(p)}_{\mu}T^{\mu}{}_{\nu} = \nabla^{(g)}_{\mu}T^{\mu}{}_{\nu} + \mathcal{C}^{\mu}_{\mu\lambda}T^{\lambda}{}_{\mu} - \mathcal{C}^{\lambda}_{\mu\nu}T^{\mu}{}_{\lambda}, \tag{32}$$

where

$$\mathcal{C}^{\alpha}_{\mu\nu} = \frac{1}{2}p^{\alpha\sigma}(\nabla^{(g)}_{\mu}p_{\sigma\nu} + \nabla^{(g)}_{\nu}p_{\mu\sigma} - \nabla^{(g)}_{\sigma}p_{\mu\nu}). \tag{33}$$

The metric/auxiliary metric relationship (27), the compatibility of $g_{\mu\nu}$ with $\nabla^{(g)}$, and the symmetry property of the auxiliary metric imply the coefficients can be expressed in a form that is independent of $p_{\mu\nu}$:

$$\mathcal{C}^{\alpha}_{\mu\nu} = \frac{1}{2}(\Omega^{-1})^{\alpha}{}_{\sigma}(\nabla^{(g)}_{\mu}\Omega^{\sigma}{}_{\nu} + \nabla^{(g)}_{\nu}\Omega^{\sigma}{}_{\mu}) - \frac{1}{2}g_{\mu\lambda}(\Omega^{-1})^{\alpha}{}_{\sigma}\nabla^{\sigma}_{(g)}\Omega^{\lambda}{}_{\nu}. \tag{34}$$

We note that any covariant derivative with respect to $\Omega^{\mu}{}_{\nu}$ can be replaced by a derivative with respect to $(\Omega^{-1})^{\mu}{}_{\nu}$, as their inverse relationship implies

$$(\Omega^{-1})^{\lambda}{}_{\nu}\nabla^{(g)}_{\sigma}\Omega^{\mu}{}_{\lambda} + \Omega^{\mu}{}_{\lambda}\nabla^{(g)}_{\sigma}(\Omega^{-1})^{\lambda}{}_{\nu} = 0. \tag{35}$$

With this in mind, the coefficients (34) become

$$\mathcal{C}^{\alpha}_{\mu\nu} = -\frac{1}{2}\Omega^{\lambda}{}_{\nu}\nabla^{(g)}_{\mu}(\Omega^{-1})^{\alpha}{}_{\lambda} - \frac{1}{2}\Omega^{\lambda}{}_{\mu}\nabla^{(g)}_{\nu}(\Omega^{-1})^{\alpha}{}_{\lambda} + \frac{1}{2}g_{\mu\lambda}(\Omega^{-1})^{\alpha}{}_{\sigma}\Omega^{\gamma}{}_{\nu}\Omega^{\lambda}{}_{\epsilon}\nabla^{\sigma}_{(g)}(\Omega^{-1})^{\epsilon}{}_{\lambda}, \tag{36}$$

and the nonconservation of the energy-momentum tensor turns out to be

$$\nabla^{(g)}_{\mu}T^{\mu}{}_{\nu} = \frac{1}{\kappa}\sqrt{\frac{p}{g}}\left\{\nabla^{(p)}_{\mu}\left[\sqrt{\frac{g}{p}}\left(f_{\mathcal{L}}\Xi^{\mu}{}_{\nu} + f_V\Pi^{\mu}{}_{\nu} - \frac{f_V}{f_{\mathcal{R}}}T^{\mu\lambda}\Sigma_{\lambda\nu} - \frac{ff_V}{2f_{\mathcal{R}}}T^{\mu}{}_{\nu}\right)\right] - \frac{1}{2}\partial_{\nu}\left(\sqrt{\frac{g}{p}}[f - \mathcal{R}f_{\mathcal{R}} - Vf_V]\right)\right\} + \mathcal{C}^{\lambda}_{\mu\nu}T^{\mu}{}_{\lambda}. \tag{37}$$

Clearly, the curvature and energy-momentum dependences in Eq. (37) and the energy-momentum dependence of $(\Omega^{-1})^{\mu}{}_{\nu}$ restrict these formulas from simplifying much beyond what is given here. We emphasize, therefore, that $\nabla^{(g)}_{\mu}T^{\mu}{}_{\nu}$ does not in general vanish. Hence, the energy-momentum tensor in Palatini $f(\mathcal{R}, \mathcal{L}_m, V)$ gravity is in general not conserved. On the other hand, in Sec. VI, we shall indirectly derive two nontrivial functions of

$f(\mathcal{R}, \mathcal{L}_m, V)$ for which the covariant divergence of specific but nontrivial $T_{\mu\nu}$ necessarily vanish, hence conserving the energy-momentum tensor. That said, this conservation will not be obvious at the level of Eq. (37), though it will nevertheless be true. Finally, we note that for the Einstein-Hilbert model $f(\mathcal{R}, \mathcal{L}_m, V) = \mathcal{R} - 2\Lambda$ the conservation of $T_{\mu\nu}$ is restored, as desired.

B. Motion of test particles

For clarity, we denote by Δ_ν the right-hand side of Eq. (37). Then, $\nabla^{(g)}_\mu T^\mu{}_\nu = \Delta_\nu$. For the case of a perfect fluid, for which $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}$, it is straightforward to show, using the constraint from the conservation of the matter fluid current, $\nabla^{(g)}_\mu(\rho u^\mu) = 0$, that

$$u^\mu \nabla^{(g)}_\mu u^\nu = \frac{\Delta^\nu - u^\nu \nabla^{(g)}_\mu (Pu^\mu) - \partial^\nu P}{P + \rho}. \tag{38}$$

Here, the left-hand side coincides with the well-known identity

$$u^\mu \nabla^{(g)}_\mu u^\nu = \frac{d^2x^\nu}{ds^2} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}. \tag{39}$$

Therefore, Eq. (38) is the equation of motion for particles in the presence of an isotropic pressure P. Absent this pressure, the equation reduces to

$$\frac{d^2x^\nu}{ds^2} + \Gamma^\nu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = f^\nu, \tag{40}$$

where the extra force $f^\nu = \rho^{-1}\Delta^\nu_{(P=0)}$ with

$$\begin{aligned}\Delta^{(P=0)}_\nu = \frac{1}{\kappa}\sqrt{\frac{p}{g}}\Bigg\{\nabla^{(p)}_\mu\Bigg[&-\sqrt{\frac{g}{p}}\frac{\rho}{2}f_{\mathcal{L}}u^\mu u_\nu + \frac{f_V\rho}{2}\sqrt{\frac{g}{p}}([\mathcal{R}^{\sigma\mu}u_\nu + \mathcal{R}^\sigma{}_\nu u^\mu]u_\sigma + \mathcal{R}^{\alpha\beta}u_\alpha u_\beta[\delta^\mu{}_\nu - 2u^\mu u_\nu] - \mathcal{R}u^\mu u_\nu)\\ &+\sqrt{\frac{g}{p}}\frac{f_V\rho^2}{\kappa f_{\mathcal{R}}}\left([\kappa - f_{\mathcal{L}}]u^\mu u_\nu + \frac{f_V}{2}[\mathcal{R}u^\mu u_\nu - u^\mu \mathcal{R}^\alpha{}_\nu u_\alpha + 4\mathcal{R}^{\alpha\beta}u_\alpha u_\beta u^\mu u_\nu]\right)\\ &-\sqrt{\frac{g}{p}}\frac{f f_V}{2f_{\mathcal{R}}}\rho u^\mu u_\nu\Bigg] - \frac{1}{2}\partial_\nu\left(\sqrt{\frac{g}{p}}[f - \mathcal{R}f_{\mathcal{R}} - Vf_V]\right)\Bigg\} + \rho \mathcal{C}^\alpha_{\beta\nu}u^\beta u_\alpha\end{aligned} \tag{41}$$

[see Eqs. (25) and (26) to derive this]. Since $\Delta^\nu_{(P=0)}$ is in general nonzero, the extra force f^ν is in general nonzero. Hence, test particles in $f(\mathcal{R}, \mathcal{L}_m, V)$ gravity do not in general obey the geodesic equation. In other words, test particles traverse geodesics of $g_{\mu\nu}$ if and only if $\Delta^\mu_{(P=0)} = 0$.

C. Newtonian limit

In the weak-field regime, we consider the gravitational effect of nonrelativistic dust, for which $T_{\mu\nu} = \rho u_\mu u_\nu$ where $u^\mu = (\partial_0)^\mu$ is the rest frame 4-velocity and ρ is the dust's energy density [38]. We shall linearize the $f(\mathcal{R}, \mathcal{L}_m, V)$ equations by keeping terms linear in ρ and in the perturbations introduced below. To facilitate the coming analysis, we adopt the following notation.

Let $\gamma_{\mu\nu}$ and $\hat{\gamma}_{\mu\nu}$ be smooth 2-forms (soon to be perturbations). Further, let $\mathcal{A}$ and $\mathcal{B}$ be mathematical objects composed, in some acceptable fashion, of the objects $\rho, \gamma_{\mu\nu}$, and $\hat{\gamma}_{\mu\nu}$. Then, by $\mathcal{A} \ll \mathcal{B}$, we shall mean $\mathcal{A}$ is first order (linear) in at least one of ρ, $\gamma_{\mu\nu}$, or $\hat{\gamma}_{\mu\nu}$, while $\mathcal{B}$ is zeroth order in all. Moreover, by $\mathcal{A} \cong \mathcal{B}$, we shall mean $\mathcal{A} = \mathcal{B}$ up to at least linear corrections in all $\rho, \gamma_{\mu\nu}$, and $\hat{\gamma}_{\mu\nu}$. Finally, by $\mathcal{A} \sim \mathcal{B}$, we shall mean $\mathcal{A}$ and $\mathcal{B}$ are of the same order in ρ, $\gamma_{\mu\nu}$, or $\hat{\gamma}_{\mu\nu}$, but not necessarily equal (thus, $\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{A} \sim \mathcal{B}$).

Consider the metric/auxiliary metric relation posited in Eq. (27). This establishes that any perturbation $\delta p_{\mu\nu}$ upon $p_{\mu\nu}$ satisfies

$$\delta p_{\mu\nu} = g_{\mu\lambda}\delta\Omega^\lambda{}_\nu + \Omega^\lambda{}_\nu \delta g_{\mu\lambda}. \tag{42}$$

Specifically, we shall consider perturbations $\delta p_{\mu\nu} = \hat{\gamma}_{\mu\nu}$ and $\delta g_{\mu\nu} = \gamma_{\mu\nu}$ upon a Minkowski background $\eta_{\mu\nu}$. Then, $p_{\mu\nu} = \eta_{\mu\nu} + \hat{\gamma}_{\mu\nu}$ and $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$ such that $\hat{\gamma}_{\mu\nu}, \gamma_{\mu\nu} \ll \eta_{\mu\nu}$. Here, $p_{\mu\nu}$ and $g_{\mu\nu}$ are related by Eq. (27), and furthermore, the perturbations $\hat{\gamma}_{\mu\nu}$ and $\gamma_{\mu\nu}$ satisfy Eq. (42). Together, these imply $\eta_{\mu\lambda}\Omega^\lambda{}_\nu - \eta_{\mu\nu} \ll \eta_{\mu\nu}$, which is possible if and only if $\Omega^\mu{}_\nu \cong \delta^\mu{}_\nu + k^\mu{}_\nu$, where $k^\mu{}_\nu \ll \delta^\mu{}_\nu$. In deriving this, one must also assume that the deformation matrix reacts smoothly to slight perturbations upon the Minkowski background, i.e., that $g_{\mu\lambda}\delta\Omega^\lambda{}_\nu \ll \eta_{\mu\nu}$, which necessitates $\eta_{\mu\lambda}\delta\Omega^\lambda{}_\nu \sim \gamma_{\mu\nu}$ and $\gamma_{\mu\lambda}\delta\Omega^\lambda{}_\nu \cong 0$.

Transcribed to matrix notation, we have $\mathbf{\Omega} \cong \mathbf{I} + \mathbf{k}$. Thus, $\mathbf{\Omega}^{-1} \cong \mathbf{I} - \mathbf{k}$, to which we shall directly compare Eq. (28). Since the tensor $T^\mu{}_\nu \to \mathbf{g}^{-1}\mathbf{T}$ is in general different from the identity matrix, it must be that $f_{\mathcal{R}} \cong \sqrt{\Omega}$ and $f_V \mathbf{g}^{-1}\mathbf{T} \cong \sqrt{\Omega}\mathbf{k}$, where $\sqrt{\Omega} \cong 1 + \frac{1}{2}\mathrm{Tr}(\mathbf{k})$. But since $\mathrm{Tr}(\mathbf{k})\mathbf{k} \cong \mathbf{0}$, with 0 the zero matrix, we simply have $\mathbf{k} \cong f_V \mathbf{g}^{-1}\mathbf{T}$. Together, these results yield both $p_{\mu\nu}$ and $p^{\mu\nu}$ to the desired first-order precision:

$$p_{\mu\nu} \cong \eta_{\mu\nu} + \gamma_{\mu\nu} + f_V T_{\mu\nu}, \tag{43a}$$

$$p^{\mu\nu} \cong \eta^{\mu\nu} - \gamma^{\mu\nu} - f_V T^{\mu\nu}, \tag{43b}$$

where $T^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}T_{\alpha\beta}$ and $\gamma^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}\gamma_{\alpha\beta}$. We see from Eq. (43a) that

$$\hat{\gamma}_{\mu\nu} \cong \gamma_{\mu\nu} + f_V T_{\mu\nu} \tag{44}$$

and similarly from Eq. (43b) that $\hat{\gamma}^{\mu\nu} \cong \gamma^{\mu\nu} + f_V T^{\mu\nu}$. Hence, the connection coefficients (15) are, to linear order in $\hat{\gamma}_{\mu\nu}$,

$$\Gamma^{\alpha}_{\mu\nu} \cong \frac{1}{2}\eta^{\alpha\sigma}(\partial_\mu \hat{\gamma}_{\sigma\nu} + \partial_\nu \hat{\gamma}_{\mu\sigma} - \partial_\sigma \hat{\gamma}_{\mu\nu}), \tag{45}$$

which yields the Ricci tensor

$$\mathcal{R}_{\mu\nu} \cong \partial^\sigma \partial_{(\nu}\hat{\gamma}_{\mu)\sigma} - \frac{1}{2}\partial^\sigma\partial_\sigma \hat{\gamma}_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu \hat{\gamma}, \tag{46}$$

where $\hat{\gamma} \equiv \hat{\gamma}^\mu{}_\mu$. Note that Eq. (46) is entirely of linear order in $\hat{\gamma}_{\mu\nu}$. Hence, $V = \mathcal{R}_{\mu\nu}T^{\mu\nu} \cong 0$ since $T^{\mu\nu}$ is linear in ρ. Moreover, $f_V \Pi_{\mu\nu} \cong 0$ since $\delta T^{\alpha\beta}/\delta g^{\mu\nu} \sim \rho$ for dust [see Eq. (24)], and $\mathcal{R}_{\mu\nu} \sim \hat{\gamma}_{\mu\nu}$. We shall impose the Lorenz gauge $\partial^\sigma \hat{\gamma}_{\mu\sigma} = 0$ so that the $f(\mathcal{R}, \mathcal{L}_m, V)$ field equations, to linear order in $\hat{\gamma}_{\mu\nu}$ and ρ, bear the form

$$-\frac{1}{2}\partial^\sigma\partial_\sigma \hat{\gamma}_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu \hat{\gamma} - \frac{1}{2} f g_{\mu\nu} \cong \kappa T_{\mu\nu} - f_{\mathcal{L}} \Xi_{\mu\nu}. \tag{47}$$

To obtain the matter tensor, we necessarily take $\mathcal{L}_m = -\rho$ for the matter Lagrangian of the pressureless dust. Hence, from Eq. (24), $\Xi_{\mu\nu} = \frac{1}{2}\rho(u_\mu u_\nu - \eta_{\mu\nu} - \gamma_{\mu\nu})$. Here, ρ is the leading-order correction from the matter sector; thus, the product $\rho\gamma_{\mu\nu}$ must be regarded as a second-order correction. This implies that, to first order, $\Xi_{\mu\nu} \cong \frac{1}{2}\rho(u_\mu u_\nu - \eta_{\mu\nu})$ and hence $\Xi \equiv \Xi^\mu{}_\mu \cong -\frac{5}{2}\rho$. It follows that the trace of the field equations (10) is, to first order,

$$f_{\mathcal{R}}\mathcal{R} - 2f \cong \rho\left(\frac{5}{2}f_{\mathcal{L}} - \kappa\right). \tag{48}$$

Note that $\Pi^\mu{}_\mu \cong 0$ since $\Pi_{\mu\nu} \cong 0$. Thus, $\Pi^\mu{}_\mu$ is absent from Eq. (48) in this approximation. Note also that the already first-order corrections ρ and $\mathcal{R}$ force f to be at least a first-order correction; hence, we can rewrite Eq. (47) as

$$-\frac{1}{2}\partial^\sigma\partial_\sigma \hat{\gamma}_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu \hat{\gamma} - \frac{1}{2} f \eta_{\mu\nu} \cong \kappa\rho u_\mu u_\nu - \frac{1}{2}\rho f_{\mathcal{L}}(u_\mu u_\nu - \eta_{\mu\nu}). \tag{49}$$

The 00 component of this equation encodes the weak-field dynamics in which we are interested. Since the spacetime is assumed static, the time derivatives vanish, leaving the expression

$$-\frac{1}{2}\Delta\hat{\gamma}_{00} \cong \kappa\rho - \frac{1}{2}f - \rho f_{\mathcal{L}}, \tag{50}$$

where $\Delta \equiv \nabla^2$ is the Laplacian operator. Using Eq. (44) and the definition of the Newtonian potential, $\Phi \equiv -\frac{\gamma_{00}}{4}$, we obtain the modified Poisson equation in $f(\mathcal{R}, \mathcal{L}_m, V)$ gravity:

$$\Delta\Phi \cong \frac{1}{2}\kappa\rho - \frac{1}{4}f - \frac{1}{2}\rho f_{\mathcal{L}} + \frac{1}{4}\Delta(f_V \rho). \tag{51}$$

Since f is implicitly a function of $T_{\mu\nu}$, and hence of ρ, the quantity $\frac{1}{2}\kappa\rho - \frac{1}{4}f - \frac{1}{2}\rho f_{\mathcal{L}}$ acts as a sort of effective density $\frac{1}{2}\kappa\bar{\rho}$. In these terms, Eq. (51) reads

$$\Delta\Phi \cong \frac{1}{2}\kappa\bar{\rho} + \frac{1}{4}\Delta(f_V\rho). \tag{52}$$

This modification to Poisson's equation is formally identical to those in both EiBI and Palatini $f(\mathcal{R}, T)$ gravity (see Refs. [26,28], respectively). Consequently, we expect all these theories to afford similar nonrelativistic phenomenology.

V. SOME APPLICATIONS

The weak-field equations considered above disclosed a relationship between $f(\mathcal{R}, \mathcal{L}_m, V)$ and other theories of gravity in the Newtonian regime. In this section, we derive the field equations governing the response of $f(\mathcal{R}, \mathcal{L}_m, V)$ gravity in other regimes, in particular the electromagnetic and scalar field sectors.

A. Electromagnetic fields

Consider next the traditional linear electrodynamics (LED) of Maxwell for which the matter Lagrangian is $\mathcal{L}^{(\mathrm{LED})} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor. The LED matter tensor follows quickly from its definition (8), $\Xi^{(\mathrm{LED})}_{\mu\nu} = -\frac{1}{8\pi}F_{\mu\lambda}F^\lambda{}_\nu$. Alternatively, using Eq. (18),

$$\Xi^{(\mathrm{LED})}_{\mu\nu} = \frac{1}{2}(\mathcal{L}^{(\mathrm{LED})}g_{\mu\nu} - T^{(\mathrm{LED})}_{\mu\nu}). \tag{53}$$

Similarly, from the symmetry of the Ricci tensor and Eqs. (21) and (53), the LED matter-curvature tensor turns out to be

$$\Pi^{(\mathrm{LED})}_{\mu\nu} = 2\mathcal{R}^\lambda{}_{(\mu}T^{(\mathrm{LED})}_{\nu)\lambda} - \mathcal{R}_{\mu\nu}\mathcal{L}^{(\mathrm{LED})} - \frac{1}{8\pi}\mathcal{R}F_{\mu\lambda}F^\lambda{}_\nu + \frac{1}{4\pi}\mathcal{R}^{\alpha\beta}F_{\mu\beta}F_{\alpha\nu}. \tag{54}$$

The LED effective energy-momentum tensor, $\Sigma^{(\mathrm{LED})}_{\mu\nu}$, then follows trivially from its definition (11).

The matter-curvature couplings in Eq. (54) are very much unlike, for instance, Palatini $f(\mathcal{R},T)$ theory, in which $T^{(\mathrm{LED})}=g^{\mu\nu}T^{(\mathrm{LED})}_{\mu\nu}=0$. Hence, the $f(\mathcal{R},T)$ models (whether Palatini or not) respond to linear electromagnetic fields as an $f(\mathcal{R})$ model. This is evidently not the case for the present theory, in which there are nontrivial couplings between curvature and matter terms, all of which have the potential to invite new gravitational electrodynamic behavior. We note that in the Palatini $f(\mathcal{R},T,V)$ model all the f_V couplings in Eq. (54) persist; therefore, even with a vanishing trace [making the gravitational response a Palatini $f(\mathcal{R},V)$ theory], there remain new and nontrivial corrections to the linear electrodynamics.

However, it is well known that the linear electrodynamics *in vacuo* are only an approximation to the full electrodynamic theory. General relativity, for example, demands a gravitational coupling between electromagnetic fields, which affords nonlinear electrodynamic behavior. That said, more considerable nonlinearity arises from quantum field effects, such as vacuum polarization [39]. It is therefore of interest to also derive the $f(\mathcal{R},\mathcal{L}_m,V)$ field equations associated with a general set of nonlinear electrodynamic (NED) theories. To this end, we set the matter sector action to

$$\mathcal{S}^{(\mathrm{NED})}=\frac{1}{8\pi}\int \mathrm{d}^4x\sqrt{-g}\chi(I,J), \tag{55}$$

where χ is a well-behaved function of the algebraic invariants $I\equiv\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $J\equiv F_{\mu\nu}(\star F)^{\mu\nu}$. Here, $(\star F)^{\mu\nu}=\frac{1}{2}(-g)^{-\frac{1}{2}}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the Hodge dual of the Faraday tensor, with $\epsilon^{\mu\nu\alpha\beta}$ denoting the Levi-Civita symbol. We note that I and J are the unique algebraic invariants constructible from $F_{\mu\nu}$ and $g_{\mu\nu}$ [40,41] and also that the choice $\chi(I,J)=-I$ corresponds to the LED theory considered above.

With $\frac{1}{8\pi}\chi(I,J)$ as the NED matter Lagrangian, and defining $\chi_I\equiv\frac{\partial\chi}{\partial I}$ and $\chi_J\equiv\frac{\partial\chi}{\partial J}$, we find

$$\Xi^{(\mathrm{NED})}_{\mu\nu}=\frac{1}{8\pi}\left(\chi_I F_{\mu\lambda}F^{\lambda}{}_{\nu}+\frac{1}{2}\chi_J J g_{\mu\nu}\right). \tag{56}$$

It then follows from Eq. (21) and the NED equivalent of Eq. (53) that

$$\begin{aligned}\Pi^{(\mathrm{NED})}_{\mu\nu}&=2\mathcal{R}^{\lambda}{}_{(\mu}T^{(\mathrm{NED})}_{\nu)\lambda}+\frac{1}{8\pi}(\chi_J J-\chi)\mathcal{R}_{\mu\nu}+\frac{1}{4\pi}\left(\frac{1}{2}\mathcal{R}\chi_I-\mathcal{R}^{\alpha\beta}F_{\alpha\lambda}F^{\lambda}{}_{\beta}\chi_{II}-\frac{1}{2}\mathcal{R}\chi_{JI}J^2\right)F_{\mu\lambda}F^{\lambda}{}_{\nu}\\&\quad-\frac{1}{8\pi}\left(\mathcal{R}^{\alpha\beta}F_{\alpha\lambda}F^{\lambda}{}_{\beta}\chi_{IJ}J+\frac{1}{2}\mathcal{R}\chi_{JJ}J^2\right)g_{\mu\nu}-\frac{1}{4\pi}\mathcal{R}^{\alpha\beta}\chi_I F_{\alpha\mu}F_{\nu\beta}.\end{aligned} \tag{57}$$

As before, $\Sigma^{(\mathrm{NED})}_{\mu\nu}$ then follows from its definition (11), and $T^{(\mathrm{NED})}_{\mu\nu}$ follows from the NED equivalent of Eq. (53). Note that, as expected, upon fixing $\chi(I,J)=-I$, Eq. (57) reduces to Eq. (54). As in the LED case, these field equations have in them nontrivial matter-curvature couplings which again bear new possibilities for NED gravitational dynamics, such as in studies of nonsingular black holes. We also note that these equations again differ drastically in their matter-curvature couplings from the field equations for NED in $f(\mathcal{R},T)$ gravity (see, e.g., Ref. [28]). This much is evident from the f_V coupling terms, which persist only in the $f(\mathcal{R},\mathcal{L}_m,V)$ framework.

B. Canonical scalar fields

Scalar fields comprise another set of generic matter fields for which $f(\mathcal{R},\mathcal{L}_m,V)$ gravity admits new and nontrivial dynamics. Here, we shall consider the effect of a real-valued scalar field ϕ in a potential $U(\phi)$, the Lagrangian density of which bears the form $\mathcal{L}^{(\phi)}=-\frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi-U(\phi)$. One shall find $\Xi^{(\phi)}_{\mu\nu}=-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi$ and, from Eq. (20),

$$\Pi^{(\phi)}_{\mu\nu}=-\frac{1}{2}\mathcal{R}\partial_\mu\phi\partial_\nu\phi+\partial_\lambda\phi\partial_{(\mu}\phi\mathcal{R}^{\lambda}{}_{\nu)}+\mathcal{R}_{\mu\nu}\mathcal{L}^{(\phi)}. \tag{58}$$

Hence,

$$\begin{aligned}\kappa\Sigma^{(\phi)}_{\mu\nu}&=\kappa T^{(\phi)}_{\mu\nu}+\frac{1}{2}(f_{\mathcal{L}}+\mathcal{R}f_V)\partial_\mu\phi\partial_\nu\phi-f_V\mathcal{L}^{(\phi)}\mathcal{R}_{\mu\nu}\\&\quad-f_V\partial_\lambda\phi\partial_{(\mu}\phi\mathcal{R}^{\lambda}{}_{\nu)}.\end{aligned} \tag{59}$$

As with the electromagnetic field, specifying particular $f(\mathcal{R},\mathcal{L}_m,V)$ functions and solving the associated field equations will conceivably yield new nonminimal corrections to ordinary GR problems, which brings about new possibilities. For example, as posited for Palatini $f(\mathcal{R},T)$ gravity [28], free $[U(\phi)=0]$ geonic solutions of the kind in EiBI gravity [42] are conceivable in the present theory.

VI. COMPATIBILITY WITH EiBI GRAVITY

In this section, we shall investigate the conditions under which the $f(\mathcal{R},\mathcal{L}_m,V)$ paradigm encapsulates the EiBI theory. We shall denote by f_{BI} any $f(\mathcal{R},\mathcal{L}_m,V)$ function that does this. To begin, it is imperative that we be precise with the meaning of "one gravitational theory corresponding to another."

Let $\mathcal{A}$ and $\mathcal{B}$ be two Palatini theories of gravity defined on a world manifold $\mathcal{M}$, and let Ψ be a matter field on $\mathcal{M}$. Further, let $g^{(\mathcal{A})}_{\mu\nu}$ and $g^{(\mathcal{B})}_{\mu\nu}$ be the solutions generated from $\mathcal{A}$ and $\mathcal{B}$, respectively, in response to Ψ, and $\nabla^{(\mathcal{A})}$ and $\nabla^{(\mathcal{B})}$ be the derivative operators of $\mathcal{A}$ and $\mathcal{B}$, respectively, defined on $\mathcal{M}$. On one hand, we say $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if, for all Ψ, (i) $\nabla^{(\mathcal{A})}_\sigma \xi^\mu = \nabla^{(\mathcal{B})}_\sigma \xi^\mu$ for all vectors ξ^μ defined on some tangent space in the tangent bundle of $\mathcal{M}$ and (ii) $g^{(\mathcal{A})}_{\mu\nu} = \Theta^2 g^{(\mathcal{B})}_{\mu\nu}$ for some real-valued, smooth conformal factor Θ defined on $\mathcal{M}$. Evidently, condition i ensures that both theories measure the same intrinsic curvature of $\mathcal{M}$, that both have the same notion of transport, and so forth, while condition ii establishes that the gravitational dynamics of the two theories are the same (since they afford the same solution, up to a conformal factor, for a given matter sector Ψ). On the other hand, we say $\mathcal{A}$ and $\mathcal{B}$ are circumstantially equivalent if conditions i and ii hold only for particular Ψ. Indeed, we shall prove in this section that EiBI and $f(\mathcal{R}, \mathcal{L}_m, V)$ are circumstantially equivalent theories of gravity; in particular, that condition i shall hold well for all Ψ but that condition ii shall hold well only for specific Ψ.

The (Palatini) EiBI action bears the form [26]

$$S_{\mathrm{BI}}[g, \Gamma, \Psi] = \frac{1}{2\kappa\epsilon}\int \mathrm{d}^4x \left[\sqrt{|g_{\mu\nu} + \epsilon\mathcal{R}_{\mu\nu}(\Gamma)|} - \lambda\sqrt{-g}\right] + S_m[g, \Psi], \tag{60}$$

where ϵ is a coupling parameter, λ is related to the cosmological constant Λ by $\lambda = 1 + \epsilon\Lambda$, and the vertical bars denote the absolute value of the determinant. The reader is referred to Refs. [25,26] for details on the variation. The field equations are

$$q_{\mu\nu} = g_{\mu\nu} + \epsilon\mathcal{R}_{\mu\nu}, \tag{61a}$$

$$\sqrt{-q}q^{\mu\nu} = \sqrt{-g}(\lambda g^{\mu\nu} - \kappa\epsilon T^{\mu\nu}), \tag{61b}$$

where q is the determinant of the auxiliary metric $q_{\mu\nu}$ and $q^{\mu\nu}$ satisfies both $q^{\mu\lambda}q_{\lambda\nu} = \delta^\mu{}_\nu$ and $\nabla^{(\mathrm{BI})}_\sigma(\sqrt{-q}q^{\mu\nu}) = 0$, where $\nabla^{(\mathrm{BI})}$ is the derivative operator associated with the Palatini EiBI theory. Hence, $\nabla^{(\mathrm{BI})}_\sigma q_{\mu\nu} = 0$.

We note that $\lambda \neq 0$ (equivalently $\Lambda \neq -\epsilon^{-1}$), for otherwise Eq. (61b) implies that *in vacuo* $\sqrt{-q}q^{\mu\nu} = 0$, which is nonsense. Moreover, with $T_{\mu\nu} = 0$ and $\lambda \neq 1$, the solutions from the two theories do not coincide; EiBI affords a de Sitter or anti-de Sitter universe, while $f(\mathcal{R}, \mathcal{L}_m, V)$ outputs Minkowski space. In speaking of a possible equivalence between the theories, it is natural to demand that at least the vacuum solutions correspond. To this end, we shall hereafter fix $\lambda = 1$, making EiBI Minkowskian *in vacuo*. Note that there is no loss of generality in doing this. Should one wish to append a cosmological constant to either theory, one would simply do so via the matter sector. We have merely "tared" the two theories at the level of their vacuum solutions.

As previously defined, $\nabla^{(p)}$ is the derivative operator associated with the Palatini $f(\mathcal{R}, \mathcal{L}_m, V)$ theory. Thus, for an EiBI/$f(\mathcal{R}, \mathcal{L}_m, V)$ equivalence to exist, condition i demands that $\nabla^{(\mathrm{BI})}_\sigma \xi^\mu = \nabla^{(p)}_\sigma \xi^\mu$ for all smooth vectors ξ^μ. This implies, in particular, that

$$\nabla^{(\mathrm{BI})}_\sigma q_{\mu\nu} = \nabla^{(p)}_\sigma q_{\mu\nu} = \nabla^{(p)}_\sigma p_{\mu\nu} = 0. \tag{62}$$

The connections of both $f(\mathcal{R}, \mathcal{L}_m, V)$ and EiBI gravity are torsion free. Hence, as required by the fundamental theorem of Riemannian geometry, Eq. (62) holds well if and only if $p_{\mu\nu} = q_{\mu\nu}$, which is true if and only if $\sqrt{-q}q^{\mu\nu} = \sqrt{-p}p^{\mu\nu}$. Therefore, the definitions (14) and (61), together with condition ii, i.e., the requisite conformal relationship $g^{(f)}_{\mu\nu} = \Theta^2 g^{(\mathrm{BI})}_{\mu\nu}$ [$g^{(f)}_{\mu\nu}$ being the solution from the $f(\mathcal{R}, \mathcal{L}_m, V)$ theory], imply (with $\lambda = 1$)

$$(1 - \Theta^2 f_\mathcal{R})g^{\mu\nu}_{(\mathrm{BI})} - (\kappa\epsilon T^{\mu\nu}_{(\mathrm{BI})} + \Theta^4 f_V T^{\mu\nu}_{(f)}) = 0, \tag{63}$$

where $T^{\mu\nu}_{(\mathrm{BI})}$ and $T^{\mu\nu}_{(f)}$ are the energy-momentum tensors of the EiBI and $f(\mathcal{R}, \mathcal{L}_m, V)$ theories, respectively, each raised by their respective metric. We cannot impose *a priori* that these energy-momentum tensors be the same since they are functions of their respective metrics. We can impose, however, that the two parenthetical terms in Eq. (63) vanish separately. This is necessarily the case if we seek generality in the matter sector, as, for instance, Eq. (63) holds *in vacuo* if and only if the two parenthetical terms vanish separately. Consequently, $\Theta^2 f_\mathcal{R} = 1$, and $\kappa\epsilon T^{\mu\nu}_{(\mathrm{BI})} = \Theta^4 f_V T^{\mu\nu}_{(f)}$. Differentiating the former with respect to $\mathcal{R}$ demands that $f_\mathcal{R}$ is constant and hence that the conformal factor Θ is constant. The same is true for the latter, where differentiation upon V implies f_V is constant.

These results indicate two things. First, $T^{\mathrm{BI}}_{\mu\nu} \propto T^{(f)}_{\mu\nu}$. For equivalence between the two theories to hold, this *constant* proportionality must hold in general, for arbitrary choices of the matter sector. But since the conformal transformation properties of the energy-momentum tensor depend on the matter sector, constant proportionality is guaranteed only with exact equality between the metrics, i.e., with $\Theta^2 = 1$ and hence $f_V = -\kappa\epsilon$. Second, the vanishing of the second derivatives $f_{\mathcal{R}\mathcal{R}}$ and f_{VV} implies that the EiBI/$f(\mathcal{R}, \mathcal{L}_m, V)$ function f_{BI} is of the form $f_{\mathrm{BI}}(\mathcal{R}, \mathcal{L}_m, V) = f_1(\mathcal{R}) + h(\mathcal{L}_m) + f_2(V)$ for well-behaved functions f_1, h, and f_2. In fact, with the conformal factor set at unity and the energy-momentum tensors identical, we simply have from Eq. (63) that $f_1(\mathcal{R}) = \mathcal{R}$ and $f_2(V) = -\kappa\epsilon V$. Hence, from Eq. (10), the f_{BI} field equations bear the form

$$\mathcal{R}_{\mu\nu}-\frac{1}{2}(\mathcal{R}+h-\kappa\epsilon V)g_{\mu\nu} = \kappa T_{\mu\nu}-h'\Xi_{\mu\nu}+2\kappa\epsilon\mathcal{R}_{\lambda(\mu}T^{\lambda}{}_{\nu)}+\kappa\epsilon\mathcal{R}^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}, \quad (64)$$

where $g_{\mu\nu}=g_{\mu\nu}^{(\mathrm{BI})}=g_{\mu\nu}^{(f)}$, $T_{\mu\nu}=T_{\mu\nu}^{(\mathrm{BI})}=T_{\mu\nu}^{(f)}$, $h'\equiv dh/d\mathcal{L}_m$, and the identity (21) has been substituted for the matter-curvature tensor.

We now wish to compare Eq. (64) with the EiBI field equations (61) to fix the function h in f_{BI}. However, at the level of the EiBI equations (61), it is not obvious how to do this. Fortunately, the EiBI equations are equivalent to the more useful form [26]

$$\epsilon\mathcal{R}_{\mu\nu}+\left(1-\sqrt{\frac{q}{g}}\right)g_{\mu\nu}=\kappa\epsilon T_{\mu\nu}+\kappa\epsilon^2\mathcal{R}_{\lambda(\mu}T^{\lambda}{}_{\nu)}. \quad (65)$$

The spacetime is $3+1$ dimensional, so $q^{\mu\nu}q_{\mu\nu}=4$. This condition allows one to explicitly solve for $\sqrt{q/g}$. Eq. (65) becomes

$$\mathcal{R}_{\mu\nu}-\frac{1}{4}(\mathcal{R}-\kappa T-\kappa\epsilon V)g_{\mu\nu}=\kappa T_{\mu\nu}+\kappa\epsilon\mathcal{R}_{\lambda(\mu}T^{\lambda}{}_{\nu)}. \quad (66)$$

We wish to investigate what must be true of h and, possibly, $\mathcal{L}_m$ such that the field equations (64) and (66) are the same. To this end, we set them equal (by solving for $\mathcal{R}_{\mu\nu}-\kappa T_{\mu\nu}$ in both), which, after tracing the 2-forms, generates the requisite condition:

$$\mathcal{R}+\kappa\epsilon\mathcal{R}^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}g^{\mu\nu}=-2h-\kappa T+h'\Xi. \quad (67)$$

There are independent ways of satisfying this equation depending on if $\Xi=0$ or $\Xi\neq 0$. Hence, one will have to choose h based on the matter sector under consideration, which demonstrates that $f(\mathcal{R},\mathcal{L}_m,V)$ gravity is at best circumstantially equivalent to the EiBI framework.

For the former, we assume $\Xi=0$ identically. Then, the matter sector is constant throughout $\mathcal{M}$, implying $\mathcal{L}_m=\Lambda/\kappa$. In this regime, EiBI gravity is known to produce a de Sitter/anti-de Sitter universe equivalent to GR [25]. Therefore, $\mathcal{R}=-4\Lambda$, and so, in order for $f(\mathcal{R},\mathcal{L}_m,V)$ theory to match EiBI theory, one ultimately demands from Eq. (67) that $h=-2\epsilon\Lambda^2$ so that

$$f_{\mathrm{BI}}(\mathcal{R},\mathcal{L}_m,V)=\mathcal{R}-2\epsilon\Lambda^2-\kappa\epsilon V. \quad (68)$$

This solution implies that $f(\mathcal{R},\mathcal{L}_m,V)$ gravity can be nontrivially fashioned to have the same de Sitter/anti-de Sitter solutions as both EiBI and GR. It also bespeaks a degeneracy in the $f(\mathcal{R},\mathcal{L}_m,V)$ framework since the independent ("trivial") choice $f(\mathcal{R},\mathcal{L}_m,V)=\mathcal{R}-2\Lambda$ would just as well deliver the de Sitter/anti-de Sitter spacetime.

For $\Xi\neq 0$, the process of choosing an h is not as straightforward. We do so in a fashion that shall let us get rid of constraints on curvature. We note, however, that one could in principle impose constraints on curvature to generate more solutions. In our approach, we shall keep in mind two things. First, h is only a function of the matter Lagrangian density. No terms involving curvature may be appear in its differential equation. Second, not imposing constraints on curvature implies the curvature terms should cancel themselves due to a judicious choice of the matter sector. There is a unique prescription that satisfies these conditions—namely, that h which makes the right side of Eq. (67) vanish identically and the corresponding $\mathcal{L}_m$ that makes the left side follow suit.

Demanding the right side of Eq. (67) to vanish implies the differential equation $-2h-\kappa T+h'\Xi=0$. We shall impose Ξ to be nonzero identically, so that one can solve for $h(\mathcal{L}_m)$ explicitly,

$$h(\mathcal{L}_m)=\frac{1}{\omega}\left(C+\kappa\int \mathrm{d}\mathcal{L}_m\omega T\right), \quad (69)$$

where C is a constant and ω is an integrating factor given by $\omega(\mathcal{L}_m)=\exp\left(-2\int \mathrm{d}\mathcal{L}_m\Xi^{-1}\right)$. The fact that C is arbitrary implies there is not a unique f_{BI} when $\Xi\neq 0$ but rather a class of functions for which this particular EiBI/$f(\mathcal{R},\mathcal{L}_m,V)$ concordance holds well. Now, with the specific choice (69) in hand, we require from Eq. (67) that $\mathcal{R}+\kappa\epsilon\mathcal{R}^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}g^{\mu\nu}=0$. This is manifestly true in vacuum (with $\Lambda=0$). Outside vacuum, the condition simplifies by noting $\mathcal{R}_{\mu\nu}\neq 0$ and $\mathcal{R}=\mathcal{R}^{\mu\nu}g_{\mu\nu}$. Hence, by retracting the Ricci tensor, the previous condition necessitates $g_{\mu\nu}+\kappa\epsilon\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}}g^{\alpha\beta}=0$ for nonzero $\mathcal{R}_{\mu\nu}$. Tracing this expression, and applying algebra upon the identity (17), recasts the condition into a form in terms of the matter Lagrangian and trace T of the energy-momentum tensor,

$$g^{\mu\nu}g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}-2\mathcal{L}_m+T-\frac{2}{\kappa\epsilon}=0, \quad (70)$$

or exclusively in terms of the matter Lagrangian:

$$g^{\mu\nu}g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}-2g^{\mu\nu}\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}+2\mathcal{L}_m-\frac{2}{\kappa\epsilon}=0. \quad (71)$$

Consequently, any matter Lagrangian density $\mathcal{L}_m$ satisfying Eq. (71) and for which $\Xi_{\mu\nu}\neq 0$ will, in the $f(\mathcal{R},\mathcal{L}_m,V)=\mathcal{R}+h(\mathcal{L}_m)-\kappa\epsilon V$ framework [with $h(\mathcal{L}_m)$ set by Eq. (69)], spawn a gravitational response identical to that in EiBI theory. Incidentally, this implies that the nonconservation equation (37) necessarily vanishes. This follows because EiBI is a minimally coupled theory, as evident from its action (60); hence, the metric connection conserves the energy-momentum tensor. Of course, Eq. (37) also vanishes for the nontrivial de Sitter/anti-de Sitter solution (68).

VII. CONCLUSIONS

In this paper, we have investigated a union of the $f(\mathcal{R}, \mathcal{L}_m)$ and $f(\mathcal{R}, T, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity models in which we allowed arbitrary coupling between the scalar curvature, matter Lagrangian density, and a matter-curvature scalar $V \equiv \mathcal{R}_{\mu\nu}T^{\mu\nu}$. The model was studied under the Palatini formalism to generate a bimetric structure commensurate with EiBI theory. This implies, in particular, that the independent connection is the Levi-Civita connection of an energy momentum–dependent auxiliary metric that is related to the spacetime metric via a matrix transformation. The equations of motion were derived and expressed in a manner formally equivalent to $f(\mathcal{R})$ theories, following the definition of an effective energy-momentum tensor. We briefly described how one obtains the Palatini $f(\mathcal{R}, T, V)$ theory from the present theory, though the exact details of Palatini $f(\mathcal{R}, T, V)$ gravity warrant further investigation. It is of interest to better understand the extra d.o.f. that our theory possesses; for instance, if these additional d.o.f. can be interpreted as a perfect fluid entering into the dynamics (see, e.g., Ref. [43]).

The field equations impose the nonconservation of the energy-momentum tensor, which gives rise to nongeodesic motion of massive test particles via the appearance of an extra force that will have a nontrivial impact on the physics for compact objects and relativistic stars. This is like EiBI gravity, where the nontrivial matter-curvature couplings give rise to new dynamics surrounding the early Universe and black holes. In the nonrelativistic regime, the dynamics of Palatini $f(\mathcal{R}, \mathcal{L}_m, V)$ gravity are qualitatively similar to the Palatini $f(\mathcal{R}, T)$ and EiBI theories. We therefore expect all these theories to afford analogous nonrelativistic phenomenology.

With the theory's basic framework established, we introduced the primary elements for some applications. In the case of perfect fluids, the hydrodynamic field equations are nontrivially altered by the nonminimal matter-curvature couplings, even in the nonrelativistic regime. When coupled to electromagnetic fields, either the linear or nonlinear paradigms, the equations have new and nontrivial couplings, and in the case of $f(\mathcal{R}, T, V)$ theory, the electrodynamics reduce to a Palatini $f(\mathcal{R}, V)$ theory due to the vanishing trace. In this realm, the problem of nonsingular black holes can be studied from a separate perspective. Similar remarks apply to scalar fields.

The resemblance to EiBI gravity was then discussed. We showed that $f(\mathcal{R}, \mathcal{L}_m, V)$ gravity is circumstantially equivalent to EiBI, meaning that the two theories have identical spacetime structure and afford identical gravitational dynamics, but only in response to very specific matter fields. It is a curiosity if the conformal transformation properties of both EiBI gravity and $f(\mathcal{R}, \mathcal{L}_m, V)$ gravity behave similarly in the matter sector in which they are known to yield identical gravitational dynamics. (For general information on the conformal transformation properties of many extended theories of gravity, see Ref. [44].) In this regard, the conformal invariance of both theories can be better understood, potentially spawning new domains in which to study AdS/CFT dualities (see, e.g., Ref. [45]).

In summary, the Palatini $f(\mathcal{R}, \mathcal{L}_m, \mathcal{R}_{\mu\nu}T^{\mu\nu})$ gravity theory considered in this work generates a myriad of avenues for future research and the potential to explore new physics. Further research is expected in this respect, on which we hope to report soon.

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