A Combinatorial LSB Theorem on the *d*-Cube

Aaron Mazel-Gee, Kyle E. Kinneberg, Mutiara Sondjaja, and Francis E. Su

Background.
The Lusternik-Schnirelman-Borsuk (LSB) theorem is a statement about covering the d-sphere with $d+1$ sets. It states the following:

Theorem 1. (Lusternik-Schnirelman-Borsuk, 1929) Suppose the d-dimensional sphere is covered by $d + 1$ closed sets. Then at least one set contains antipodal points.

Results.
We prove an analogue of this theorem for ridge coverings of C^d, a d-dimensional cube:

Theorem 2. (Mazel-Gee, Kinneberg, Sondjaja, Su) Suppose the $(d-2)$-dimensional faces of C^d are covered by d sets. Then at least one set contains be a pair of antipodal k-dimensional faces, where $k = d - 2$ for $d = 1$; $k = d - 3$ for $d = 2, 3, 4$; and $k = d - 4$ for $d \geq 5$. For all dimensions except possibly $d = 5$, this k is best possible.

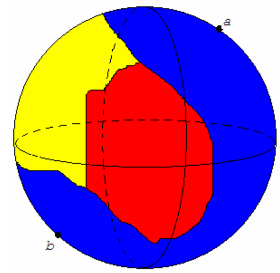

Figure 1. An illustration of the LSB theorem on a *2*-sphere. Points *a* and *b* on the blue set are antipodal to each other.

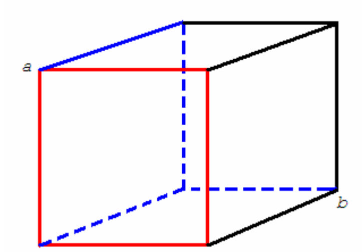

Figure 2. An illustration of Theorem 2 on a *3*-cube. We cover the *1*-faces (edges) of the *3*-cube with three sets. The *0*-faces (vertices) *a* and *b* on the blue set are antipodal to each other.

Background.
The Kakutani fixed point theorem is a generalization of a more basic fixed point theorem by Brouwer (1912), which is equivalent to a combinatorial result called Sperner's Lemma. We wish to prove Kakutani's theorem directly from Sperner's lemma.

Theorem 1. (Kakutani Fixed Point Theorem, 1942) If S is an n-dimensional closed simplex and f is an upper semicontinuous point-to-set mapping from S to convex and compact subsets of S, then $f(x)$ has a fixed point.

Theorem 2. (Sperner's Lemma) Let S be an n-dimensional simplex with vertices labeled $0, \ldots, n$. If S is triangulated and is given a proper Sperner labeling, then there exists a completely labeled subsimplex.

Results.
We prove a weaker version of Kakutani's theorem using Sperner's lemma. However, a proof of the original theorem from the lemma still remains to be done.

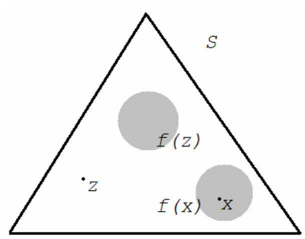

Figure 1. An illustration of the Kakutani fixed point theorem. Here, the point is contained in its own image ($f(x)$), hence is a fixed point of the mapping f.

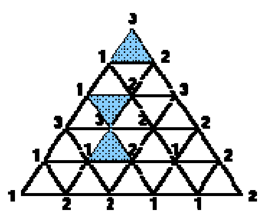

Figure 2. An illustration of Sperner's lemma. Dividing the large triangle into smaller ones and labeling the vertices with *1*, *2*, or *3*, there exists at least one small triangle with three distinct labels (shaded blue)

Funding: Beckman Research grant